MATH 124B: HOMEWORK 2

Suggested due date: August 15th, 2016

(1) Consider the geometric series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

(a) Does it converge pointwise in the interval $-1 < x < 1$?

(b) Does it converge uniformly in the interval $-1 < x < 1$?

(c) Does it converge in the L^2 sense in the interval $-1 < x < 1$?

(2) Let $f(x)$ be a function on $(-L, L)$ that has a continuous derivative and satisfies the periodic boundary conditions. Let a_n be the Fourier cosine coefficients and b_n be the Fourier sine coefficients of $f(x)$ and let a_n' and b_n' be the corresponding Fourier coefficients of its derivative $f'(x)$. Show that

$$a_n' = \frac{n\pi b_n}{L} \text{ and } b_n' = \frac{-n\pi a_n}{L} \text{ for } n \neq 0.$$

Deduce from this that there is a constant k independent of n such that

$$|a_n| + |b_n| \leq \frac{k}{n} \text{ for all } n.$$

Note, this does not mean that the differentiated series converges.

(3) If $f(x)$ is a piecewise continuous function in $[-L, L]$, show that its indefinite integral $F(x) = \int_{-L}^{x} f(s)ds$ has a Full Fourier series that converges pointwise.

(4) Write this convergent series for $f(x)$ explicitly in terms of the Fourier coefficients a_0, a_n and b_n of $f(x)$. Why does this imply that we can integrate the terms of the Fourier series term by term?

(5) Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

(6) Prove the inequality $L\int_0^L (f'(x))^2 dx \geq (f(L) - f(0))^2$, for any real function $f(x)$ whose derivative $f'(x)$ is continuous.

(7) Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ that satisfies the periodic boundary condition and if $\int_{-\pi}^{\pi} f(x) = 0$, then

$$\int_{-\pi}^{\pi} |f|^2 dx \leq \int_{-\pi}^{\pi} |f'|^2 dx.$$

This inequality is known as Wirtinger's inequality and is used in the proof of the isoperimetric inequality.

Solutions

1. **a**. We will show that

$$\sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}.$$

Using the formula

$$\sum_{n=0}^{N} t^n = \frac{1-t^{N+1}}{1-t}$$

we have

$$\sum_{n=0}^{N}(-x^2)^n = \frac{1-(-x^2)^{N+1}}{1+x^2}.$$

Hence,

$$\left|\sum_{n=0}^{N}(-x^2)^n - \frac{1}{1+x^2}\right| = \frac{x^{2N+2}}{1+x^2}.$$

Since $|x| \leq 1$, the right hand side goes to 0 as $N \to \infty$.

b. We have

$$\sup_{[-1,1]}\left|\sum_{n=0}^{N}(-x^2)^n - \frac{1}{1+x^2}\right| \geq \frac{1}{2}$$

(why does the above hold independent of N?) hence does not converge uniformly.

c. The L^2 norm is

$$\int_{-1}^{1}\frac{x^{4N+4}}{(1+x^2)^2}dx \leq 2\int_{0}^{1}x^{4N+4}dx = \frac{2}{4N+5} \to 0$$

as $N \to \infty$. (Why does the first inequality hold?)

2. The coefficients are given by

$$\begin{cases} a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx \end{cases}$$

and

$$\begin{cases} a_n' = \frac{1}{L}\int_{-L}^{L} f'(x)\cos(\frac{n\pi}{L}x)dx \\ b_n' = \frac{1}{L}\int_{-L}^{L} f'(x)\sin(\frac{n\pi}{L}x)dx \end{cases}.$$

Integrating by parts,

$$\begin{aligned} a_n' &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos(\frac{n\pi}{L}x)dx \\ &= -\frac{1}{L}\int_{-L}^{L} f(x)(\cos(\frac{n\pi}{L}x))'dx \\ &= \frac{n\pi}{L^2}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx \\ &= \frac{n\pi}{L}b_n \end{aligned}$$

(why does the boundary term in the integration by parts vanish?) Similar for the other case.

Therefore, we have

$$|a_n| + |b_n| = \frac{L}{n\pi}(|a_n|' + |b_n|')$$

Now

$$\begin{aligned} |a_n'| &= \left| \frac{1}{L} \int_{-L}^{L} f'(x) \cos(\frac{n\pi}{L}x) dx \right| \\ &\leq \frac{1}{L} \int_{-L}^{L} |f'(x)| dx < \infty \end{aligned}$$

hence we obtain a constant independent of n.

3. Since $F(x)$ is differentiable, with $F'(x) = f(x)$ piecewise continuous, we can apply the pointwise convergence theorem for classical Fourier series.

4. Let A_n, B_n be the Fourier coefficients of $F(x)$.

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^{L} F(x) \cos(\frac{n\pi}{L}x) dx \\ &= \frac{1}{n\pi} \int_{-L}^{L} \left(\int_{-L}^{x} f(s) ds \right) d(\sin(\frac{n\pi}{L}x)) \\ &= -\frac{1}{n\pi} \int_{-L}^{L} f(x) \sin(\frac{n\pi}{L}x) dx \\ &= -\frac{L}{n\pi} b_n \end{aligned}$$

(why does the boundary term for integration by parts vanish?) and similarly we can show

$$B_n = \frac{L}{n\pi} a_n$$

Therefore

$$\begin{aligned} F(x) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x) + B_n \sin(\frac{n\pi}{L}x) \\ &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} -\frac{L}{n\pi} b_n \cos(\frac{n\pi}{L}x) + \frac{L}{n\pi} a_n \sin(\frac{n\pi}{L}x). \end{aligned}$$

now if we formally integrate f, assuming that $a_0 = 0$. Then we have,

$$\int_{-L}^{x} f(s) ds = \int_{-L}^{x} \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}s) + b_n \sin(\frac{n\pi}{L}s) ds$$

which equals $F(x)$ except for a constant. In fact, if $a_0 \neq 0$, then the indefinite integral is no longer a Fourier series, however the convergence of the infinite sum is guaranteed.

5. This can be done in a number of ways. We will need to use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. First compute the Fourier sine series for x^2 on the interval $(0, 1)$, which gives the coefficients

$$A_m = \begin{cases} -\frac{2}{m\pi} & m \text{ even} \\ \frac{2m^2\pi^2 - 8}{m^3\pi^3} & m \text{ odd.} \end{cases}$$

By Parseval's identity, we have

$$\sum_{m=1}^{\infty} |A_m|^2 \int_0^1 \sin^2(m\pi x) dx = \int_0^1 x^4 dx$$

Therefore,

$$\sum_{m \text{ even}} \frac{4}{m^2}\pi^2 + \sum_{m \text{ odd}} \left(\frac{4}{m^2\pi^2} - \frac{32}{m^4\pi^4} + \frac{64}{m^6\pi^6} \right) = \frac{2}{5}$$

Now the $1/m^2$ term is known and the odd part of $1/m^4$ can be computed from the whole series by

$$\sum_{\text{odd}} \frac{1}{m^4} + \sum_{m=1}^{\infty} \frac{1}{(2m)^4} = \frac{\pi^4}{90}$$

hence $\sum_{m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96}$. Hence

$$\frac{64}{\pi^6} \sum_{m \text{ odd}} \frac{1}{m^6} = \frac{1}{15}$$

or $\sum_{m \text{ odd}} \frac{1}{m^6} = \frac{\pi^6}{960}$. Since the whole series is the even and the odd terms and the even terms are $\sum_{m=1}^{\infty} \frac{1}{(2m)^6} = \frac{1}{64} \sum_{m=1} \frac{1}{m^6}$ therefore

$$\sum_{m=1}^{\infty} \frac{1}{m^6} = \frac{\pi^6}{945}.$$

6. Apply Cauchy-Schwarz with f' and 1.

7. From the assumption, we know that for the Fourier coefficient of f, $A_0 = 0$. By Parseval's equality, we have

$$\int_{-\pi}^{\pi} |f|^2 dx = \pi \sum_{n=1} (|A_n|^2 + |B_n|^2))$$

Note that $\int_{-\pi}^{\pi} X_n^2 = \pi$ for $X_n = \cos(nx)$ and $X_n = \sin(nx)$. It was shown before that $A_n = -\frac{1}{n}B_n'$ and $B_n = \frac{1}{n}A_n'$ hence

$$\pi \sum_{n=1}(|A_n|^2 + |B_n|^2)) \leq \pi \sum_{n=1}(|A_n'|^2 + |B_n'|^2)) = \int_{-\pi}^{\pi} (f')^2 dx$$

Note that $A_0' = \int_{-\pi}^{\pi} f' dx = f(\pi) - f(-\pi) = 0$ by the periodic boundary condition. (Further consideration: When is the inequality an equality? Which part of the proof will give us an idea of what type of function will be an equality? The equality case gives us a hint as to why this inequality is related to the isoperimetric inequality.)